An interesting pattern which appears in the use of periodical numbers and $\sqrt{\ }$

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1. Introduction.

In a elective class where students can use a computer algebra I made students play with numbers, and six students found a interesting patterns by themselves. They made the finding using only numbers and $\sqrt{\ }$. The fact that this kind of thing can happen is not difficult to prove, but we have not seen these patterns in books.

2. Interesting patterns we can find in calculation of periodic numbers and $\sqrt{\ }$.

Example 1. If you use periodical numbers 1212⋯12 and $\sqrt{\ }$, then you can get the following numbers. The patterns inside $\sqrt{\ }$ is interesting.

$\sqrt{12}$	$2\sqrt{3}$
$\sqrt{1212}$	$2\sqrt{303}$
$\sqrt{121212}$	$6\sqrt{3367}$
$\sqrt{12121212}$	$2\sqrt{3030303}$
$\sqrt{1212121212}$	$2\sqrt{303030303}$
$\sqrt{121212121212}$	$6\sqrt{3367003367}$
$\sqrt{12121212121212}$	$2\sqrt{3030303030303}$
$\sqrt{1212121212121212}$	$2\sqrt{303030303030303}$

Mathematica calculation for Example 1.

$$g[m\_] := 2 \sum_{k=0}^{m} 100^k + 10 \sum_{k=0}^{m} 100^k$$

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TableForm[Table[{SqrtBox[g[n]] // DisplayForm, √g[n]}, {n, 0, 7}]]
```

Example 2. If you use periodical numbers 9999⋯99 and $\sqrt{\ }$, then you can get the following numbers. Perhaps the pattern 12345679012345679 is quite familiar to someone who knows a lot about the nature of integers.

$\sqrt{9}$	3
$\sqrt{99}$	$3\sqrt{11}$
$\sqrt{999}$	$3\sqrt{111}$
$\sqrt{9999}$	$3\sqrt{1111}$
$\sqrt{99999}$	$3\sqrt{11111}$
$\sqrt{999999}$	$3\sqrt{111111}$
$\sqrt{9999999}$	$3\sqrt{1111111}$
$\sqrt{99999999}$	$3\sqrt{11111111}$
$\sqrt{999999999}$	$9\sqrt{12345679}$
$\sqrt{9999999999}$	$3\sqrt{1111111111}$

$\sqrt{99999999999}$	$3\sqrt{11111111111}$
$\sqrt{999999999999}$	$3\sqrt{111111111111}$
$\sqrt{9999999999999}$	$3\sqrt{1111111111111}$
$\sqrt{99999999999999}$	$3\sqrt{11111111111111}$
$\sqrt{999999999999999}$	$3\sqrt{111111111111111}$
$\sqrt{9999999999999999}$	$3\sqrt{1111111111111111}$
$\sqrt{99999999999999999}$	$3\sqrt{11111111111111111}$
$\sqrt{999999999999999999}$	$9\sqrt{12345679012345679}$
$\sqrt{9999999999999999999}$	$3\sqrt{1111111111111111111}$

Mathematica calculation for Example 2.

$$g[m\_] := 9 \sum_{k=0}^{m} 10^k$$

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TableForm[Table[{SqrtBox[g[n]] // DisplayForm, √g[n]}, {n, 0, 18}]]
```

Example 3. If you use periodical numbers 112112⋯112 and $\sqrt{\ }$, then you can get the following numbers.

$\sqrt{112}$	$4\sqrt{7}$
$\sqrt{112112}$	$28\sqrt{143}$
$\sqrt{112112112}$	$4\sqrt{7007007}$
$\sqrt{112112112112}$	$28\sqrt{143000143}$
$\sqrt{112112112112112}$	$4\sqrt{7007007007007}$

$\sqrt{112112112112112112}$	$28\sqrt{143000143000143}$
$\sqrt{112112112112112112112}$	$4\sqrt{7007007007007007007}$
$\sqrt{112112112112112112112112}$	$28\sqrt{143000143000143000143}$

Mathematica calculation for Example 3.

$$g[m\_] := 2\sum_{k=0}^{m} 1000^k + 10\sum_{k=0}^{m} 1000^k + 100\sum_{k=0}^{m} 1000^k$$

```
TableForm[Table[{SqrtBox[g[n]] // DisplayForm, √g[n]}, {n, 0, 7}]]
```

Example 4. If you use periodical numbers 11111⋯11 and $\sqrt{\ }$, then you can get the following numbers.

$\sqrt{11}$	$\sqrt{11}$
$\sqrt{1111}$	$\sqrt{1111}$
$\sqrt{111111}$	$\sqrt{111111}$
$\sqrt{11111111}$	$\sqrt{11111111}$
$\sqrt{1111111111}$	$\sqrt{1111111111}$
$\sqrt{111111111111}$	$\sqrt{111111111111}$
$\sqrt{11111111111111}$	$\sqrt{11111111111111}$
$\sqrt{1111111111111111}$	$\sqrt{1111111111111111}$

$\sqrt{111111111111111111}$	$3\sqrt{12345679012345679}$
$\sqrt{11111111111111111111}$	$\sqrt{11111111111111111111}$
$\sqrt{1111111111111111111111}$	$11\sqrt{9182736455463728191}$
$\sqrt{111111111111111111111111}$	$\sqrt{111111111111111111111111}$
$\sqrt{11111111111111111111111111}$	$\sqrt{11111111111111111111111111}$
$\sqrt{1111111111111111111111111111}$	$\sqrt{1111111111111111111111111111}$
$\sqrt{111111111111111111111111111111}$	$\sqrt{111111111111111111111111111111}$
$\sqrt{11111111111111111111111111111111}$	$\sqrt{11111111111111111111111111111111}$
$\sqrt{1111111111111111111111111111111111}$	$\sqrt{1111111111111111111111111111111111}$

$\sqrt{111111111111111111111111111111111111}$	$3\sqrt{12345679012345679012345679012345679}$
$\sqrt{11111111111111111111111111111111111111}$	$\sqrt{11111111111111111111111111111111111111}$
$\sqrt{1111111111111111111111111111111111111111}$	$\sqrt{1111111111111111111111111111111111111111}$
$\sqrt{111111111111111111111111111111111111111111}$	$7\sqrt{2267573696145124716553287981859410430839}$

Mathematica calculation for Example 4.

$$g[m\_] := \sum_{k=0}^{m} 100^k + 10\sum_{k=0}^{m} 100^k$$

```
TableForm[Table[{SqrtBox[g[n]] // DisplayForm, √g[n]}, {n, 0, 20}]]
```

There appear many numbers with certain patters. For example 9182736455463728191 has an interesting structure of 9+1=8+2=7+3=⋯=2+8=1+9, but there appear a number which does not have an obvious patter. For example 2267573696145124716553287981859410430839 seems to have no patter, but if we study this number carefully, we can find some interesing fact.

If you count how many times numbers 0,1,2,3,4,5,6,7,8,9 appear in the digts of 2267573696145124716553287981859410430839, then you will get the following table.

number	0	1	2	3	4	5	6	7	8	9
frequency	2	5	4	4	4	5	4	4	4	4

It surely has a kind of pattern!

Mathematica calculation for the above table.

```
cc = Table[{n, Count[IntegerDigits[226757369614512471655328798185941O430839], n]},
  {n, 0, 9}]; FrameBox[GridBox[Transpose[Join[{{number, frequency}}, cc]],
   RowLines → True, ColumnLines → True]] // DisplayForm
```

number	0	1	2	3	4	5	6	7	8	9
frequency	2	5	4	4	4	5	4	4	4	4

Example 5. If you use periodical numbers 9999⋯99 and $\sqrt{\ }$, then you can get the following numbers. Perhaps the pattern 12345679012345679 is quite familiar to someone who knows a lot about the nature of integers.

$\sqrt{16}$	4
$\sqrt{1616}$	$4\sqrt{101}$
$\sqrt{161616}$	$4\sqrt{10101}$
$\sqrt{16161616}$	$4\sqrt{1010101}$
$\sqrt{1616161616}$	$4\sqrt{101010101}$
$\sqrt{161616161616}$	$4\sqrt{10101010101}$
$\sqrt{16161616161616}$	$4\sqrt{1010101010101}$
$\sqrt{1616161616161616}$	$4\sqrt{101010101010101}$

$\sqrt{161616161616161616}$	$12\sqrt{1122334455667789}$
$\sqrt{16161616161616161616}$	$4\sqrt{1010101010101010101}$
$\sqrt{1616161616161616161616}$	$4\sqrt{101010101010101010101}$
$\sqrt{161616161616161616161616}$	$4\sqrt{10101010101010101010101}$
$\sqrt{16161616161616161616161616}$	$4\sqrt{1010101010101010101010101}$

$\sqrt{1616161616161616161616161616}$	$4\sqrt{101010101010101010101010101}$
$\sqrt{161616161616161616161616161616}$	$4\sqrt{10101010101010101010101010101}$
$\sqrt{16161616161616161616161616161616}$	$4\sqrt{1010101010101010101010101010101}$
$\sqrt{1616161616161616161616161616161616}$	$4\sqrt{101010101010101010101010101010101}$
$\sqrt{161616161616161616161616161616161616}$	$12\sqrt{11223344556677890011223344556677 89}$

Mathematica calculation for Example 5.

$$g[m\_] := 6\sum_{k=0}^{m} 100^k + 10\sum_{k=0}^{m} 100^k$$

```
TableForm[Table[{SqrtBox[g[n]] // DisplayForm, √g[n]}, {n, 0, 17}]]
```

There appears a very interesting number. 1122334455667789 has an interesting pattern.

Remark. If you want to make other patterns and enjoy studying them, it is better for you to use some computer algebra system. We used *Mathematica*, but you can use your forvorite system and some systems are free!

3. Mathematical background of these patterns.

It is not difficult to know why this kind of thing happens.

Let x be a natural number, and we denote by x_n the number xx⋯x, where we use x n-times. For example if x = 12, then x_3 = 121212. By L(x) we denote the length of digits of x. For example L(123) = 3.

Lemma 1. Let x and y be natural numbers. Suppose that y and 10 do not have a common divisor that is bigger than 1.
Then there exits a natural number n such that n ≤ y and x_n can be divided by y, and that x_{mn} can also be divided by y for any natural number m.
Proof. Let r_n be the least nonnegative residue of x_n when it is divided by y. We consider $\{r_1, r_2, \cdots r_{y+1}\}$. Since $0 \le r_n < y$ and there are y+1 elements in $\{r_1, r_2, \cdots r_{y+1}\}$, there exist natural numbers p and q such that $1 \le q < p \le y+1$ and $r_p = r_q$. Then $x_p - x_q$ can be divided by y.
Let n = p-q, then $1 \le n \le y$. $x_p - x_q = x_{p-q} \times 10^{qL(x)}$ can be divided by y. Since y and 10 do not have a common divisor bigger than 1, $x_n = x_{p-q}$ can be divided by y.
For any natural number m $x_{mn} = x_n \times 10^{(m-1)\, nL(x)} + x_n \times 10^{(m-2)\, nL(x)} + \cdots + x_n \times 10^{nL(x)} + x_n$. Since x_n can be divided by y, x_{mn} can also be divided by y for any natural number m.

Lemma 2. Let x and y be a natural number. Suppose y and 10 do not have a common divisor bigger than 1.
(1) There exist natural numbers n and z such that $\sqrt{x_n}$ can be expressed as $y\sqrt{z}$, and that
for any natural number m $\sqrt{x_{mn}}$ can be expressed as $y\sqrt{z_m}$ for some z_m.
Proof. We just have to use y^2 instead of y in Lemma 1.

Remark. Lemma 1 was proved for x = 1 in Suugaku seminar editors [1]. Lemma 1 and Lemma 2 explain the fact that patterns in Example 1,2,3,4 and 5 appear repeatedly.

Reference.

Suugaku seminar editors [1] Looking for Elegant Solutions (in Japanese),
2001 Nihon Hyoronsha.